Cycles and p-competition graphs

Suh-Ryung KIM [*] Boram PARK [†‡]

Department of Mathematics Education,
Seoul National University, Seoul 151-742, Korea.

Yoshio SANO [§]

Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan.

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Abstract

The notion of p-competition graphs of digraphs was introduced by S-R. Kim, T. A. McKee, F. R. McMorris, and F. S. Roberts [p-competition graphs, *Linear Algebra Appl.* **217** (1995) 167–178] as a generalization of the competition graphs of digraphs. Let p be a positive integer. The *p-competition graph* $C_p(D)$ of a digraph $D = (V, A)$ is a (simple undirected) graph which has the same vertex set V and has an edge between distinct vertices x and y if and only if there exist p distinct vertices $v_1, \ldots, v_p \in V$ such that $(x, v_i), (y, v_i)$ are arcs of the digraph D for each $i = 1, \ldots, p$.

In this paper, given a cycle of length n, we compute exact values of p in terms of n such that it is a p-competition graph, which generalizes the results obtained by Kim *et al.*. We also find values of p in terms of n so that its complement is a p-competition graph.

Keywords: p-competition graph; p-edge clique cover; cycle

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[†]The author was supported by Seoul Fellowship.

[‡]Corresponding author. *E-mail address:* kawa22@snu.ac.kr ; borampark22@gmail.com

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1 Introduction

Cohen [1] introduced the notion of a competition graph in connection with a problem in ecology (also see [2]). Let $D = (V, A)$ be a digraph that corresponds to a food web. A vertex in the digraph D stands for a species in the food web, and an arc $(x, v) \in A$ in D means that the species x preys on the species v. For vertices $x, v \in V$, we call v a *prey* of x if there is an arc $(x, v) \in A$. If two species x and y have a common prey v, they will compete for the prey v. Cohen defined a graph which represents the competition among the species in the food web. The *competition graph* $C(D)$ of a digraph $D = (V, A)$ is a simple undirected graph $G = (V, E)$ which has the same vertex set V and has an edge between two distinct vertices $x, y \in V$ if there exists a vertex $v \in V$ such that $(x, v), (y, v) \in A$. We say that a graph G is a *competition graph* if there exists a digraph D such that $C(D) = G$.

Kim *et al.* [5] introduced p-competition graphs as a generalization of competition graphs. Let p be a positive integer. The *p-competition graph* $C_p(D)$ of a digraph $D = (V, A)$ is a simple undirected graph $G = (V, E)$ which has same vertex set V and has an edge between two distinct vertices $x, y \in V$ if there exist p distinct vertices $v_1, ..., v_p \in V$ such that $(x, v_i), (y, v_i) \in A$ for each $i = 1, ..., p$. Note that the 1-competition graph $C_1(D)$ of a digraph D coincides with the competition graph $C(D)$ of the digraph D. A graph G is called a *p-competition graph* if there exists a digraph D such that $C_p(D) = G$.

Competition graphs are closely related to edge clique covers and the edge clique cover numbers of graphs. A *clique* of a graph G is a subset of the vertex set of G such that its induced subgraph of G is a complete graph. We regard an empty set also as a clique of G for convenience. An *edge clique cover* of a graph G is a family of cliques of G such that the endpoints of each edge of G are contained in some clique in the family. The minimum size of an edge clique cover of G is called the *edge clique cover number* of the graph G, and is denoted by $\theta_e(G)$.

Let $G = (V, E)$ be a graph and $\mathcal{F} = \{S_1, ..., S_r\}$ be a multifamily of subsets of the vertex set of G. The multifamily $\mathcal{F}$ is called a *p-edge clique cover* if it satisfies the following:

- For any $J \in \binom{[r]}{p}$, the set $\bigcap_{j\in J} S_j$ is a clique of G,
- The family $\{\bigcap_{j\in J} S_j \mid J \in \binom{[r]}{p}\}$ is an edge clique cover of G,

where $\binom{[r]}{p}$ denotes the set of all p-subsets of an r-set $[r] := \{1, 2, ..., r\}$. The minimum size of a p-edge clique cover of G is called the *p-edge clique cover number* of G, and is denoted by $\theta_e^p(G)$.

Kim *et al.* [5] gave a characterization of p-competition graphs, a sufficient condition for graphs to be p-competition graphs, and a characterization of the p-competition graphs of acyclic digraphs as follows:

Theorem 1.1 ([5])**.** *Let G be a graph with n vertices. Then G is a p-competition graph if and only if $\theta_e^p(G) \leq n$.*

Theorem 1.2 ([5])**.** *Let G be a graph with n vertices. If $\theta_e(G) \leq n-p+1$, then G is a p-competition graph.*

Theorem 1.3 ([5])**.** *Let $G = (V, E)$ be a graph with n vertices. Then, G is the p-competition graph of an acyclic digraph if and only if there exist an ordering $v_1, ..., v_n$ of the vertices of G and a p-edge clique cover $\{S_1, ..., S_n\}$ of G such that $v_i \in S_j \Rightarrow i < j$.*

In this paper, given a cycle of length n, we compute exact values of p in terms of n such that it is a p-competition graph, which generalizes the results obtained by Kim *et al.* [5]. We also find values of p in terms of n so that its complement is a p-competition graph. Throughout this paper, let C_n be a cycle with n vertices:

$$V(C_n) = \{v_0, ..., v_{n-1}\}, \quad E(C_n) = \{v_i v_{i+1} \mid i = 0, ..., n-1\},$$

and any subscripts are reduced to modulo n.

2 Main Results

Kim *et al.* [5] showed the following results:

Theorem 2.1 ([5])**.** *C_4 is not a 2-competition graph.*

Theorem 2.2 ([5])**.** *For $n > 4$, C_n is a 2-competition graph.*

The following theorem is a generalization of the above results.

Theorem 2.3. *Let C_n be a cycle with n vertices and p be a positive integer. Then C_n is a p-competition graph if and only if $n \geq p + 3$.*

Proof. First, we show the 'only if' part by contradiction. Suppose that the cycle C_n is a p-competition graph with a positive integer p such that $p \geq n - 2$. By Theorem 1.1, there exists a p-edge clique cover $\mathcal{F}$ of C_n which consists of n sets.

Suppose that v_i belongs to all the n sets in $\mathcal{F}$ for some $i \in \{0, 1, \ldots, n-1\}$. Then, since v_{i+1} and v_{i+2} are adjacent, there exist p sets in $\mathcal{F}$ which contain both v_{i+1} and v_{i+2}. Since those p sets contain v_i, the vertices v_i and v_{i+2} are adjacent, which is a contradiction. Thus, each vertex on C_n

belongs to at most $n-1$ sets in $\mathcal{F}$. Since v_i is adjacent to v_{i+1} and v_{i-1}, it must belong to at least p sets which contain v_{i+1} and at least p sets which contain v_{i-1}. On the other hand, since v_{i+1} and v_{i-1} are not adjacent, there are at most $p-1$ sets which contain both v_{i+1} and v_{i-1}. Thus v_i belongs to at least $(p+p)-(p-1)=p+1$ sets in $\mathcal{F}$. Therefore

$$p+1 \leq (\text{the number of sets in } \mathcal{F} \text{ containing } v_i) \leq n-1.$$

However, $p+1 \geq n-1$ by our assumption that $p \geq n-2$. Thus v_i belongs to exactly $n-1$ sets in $\mathcal{F}$. Since v_i was arbitrarily chosen, v_{i+2} also belongs to exactly $n-1$ sets in $\mathcal{F}$. Then v_i and v_{i+2} belong to at least $n-2$ sets together. However, since $n-2=p$, v_i and v_{i+2} are adjacent and we reach a contradiction. Hence $n \geq p+3$.

Now we show the 'if' part. Let p be a positive integer such that $p+3 \leq n$. Let

$$S_i := \{v_i, v_{i+1}, ..., v_{i+p}\} \quad (i=0,...,n-1).$$

Then we put $\mathcal{F} := \{S_0, S_1, \dots, S_{n-1}\}$. It is easy to check that for each i, j with $i<j$,

$$S_i, S_{i-1}, \dots, S_{j-p}$$

are the sets containing both v_i and v_j. Thus there are p sets in $\mathcal{F}$ containing both v_i and v_{i+1}, and there are at most $p-1$ sets in $\mathcal{F}$ containing both v_i and v_j for i, j with $j-i \geq 2$. Thus $\mathcal{F}$ is a p-edge clique cover of C_n. Hence $\theta_e^p(C_n) \leq n$. By Theorem 1.1, the cycle C_n is a p-competition graph. $\square$

In the rest of the paper, we give a sufficient condition for the complements of cycles to be p-competition graphs. We first study on the edge clique cover numbers of complements of cycles:

Theorem 2.4. *Let $\overline{C_n}$ be the complement of a cycle with n vertices. Then the following are true:*

(i) $\theta_e(\overline{C_5})=5$, $\theta_e(\overline{C_6})=5$, $\theta_e(\overline{C_7})=7$, $\theta_e(\overline{C_8})=6$.

(ii) *If n ($n \geq 9$) is odd, then $\theta_e(\overline{C_n}) \leq \frac{n+5}{2}$.*

(iii) *If n ($n \geq 10$) is even, then $\theta_e(\overline{C_n}) \leq \frac{n}{2}+1$.*

Proof. (i) Since $\overline{C_5} \cong C_5$, we have $\theta_e(\overline{C_5})=5$.

For $\overline{C_6}$, any two edges of $\{v_0v_3, v_1v_4, v_2v_5, v_1v_5, v_2v_4\}$ cannot be covered by a clique as the set of their endpoints contains two consecutive vertices of C_6. Thus $\theta_e(\overline{C_6}) \geq 5$. In addition, $\{v_0,v_2,v_4\}, \{v_1,v_3,v_5\}, \{v_2,v_5\}, \{v_1,v_4\}, \{v_0,v_3\}$ form an edge clique cover of $\overline{C_6}$, and so $\theta_e(\overline{C_6})=5$.

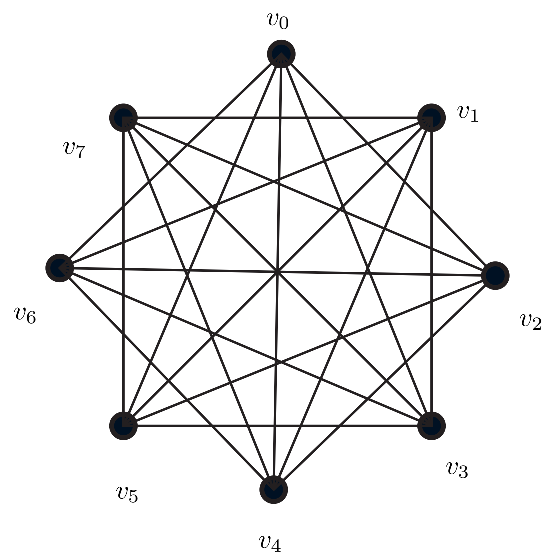

Figure 1: $\overline{C_8}$

For $\overline{C_7}$, any two edges of $\{v_0v_3, v_1v_4, v_2v_5, v_3v_6, v_4v_0, v_5v_1, v_6v_2\}$ cannot be covered by a clique and so $\theta_e(\overline{C_7}) \geq 7$. Since

$\{v_0, v_2, v_5\}, \{v_1, v_3, v_6\}, \{v_2, v_0, v_4\}, \{v_3, v_1, v_5\}, \{v_4, v_2, v_6\}, \{v_0, v_3\}, \{v_1, v_4\}$

form an edge clique cover of $\overline{C_7}$, $\theta_e(\overline{C_7}) = 7$.

For $\overline{C_8}$, any two edges of $\{v_0v_2, v_0v_3, v_1v_3, v_1v_4, v_2v_7\}$ cannot be covered by same clique and so $\theta_e(\overline{C_8}) \geq 5$ (see Figure 1 for illustration). Suppose that there is an edge clique cover of $\overline{C_8}$ whose size is 5, say $\{S_1, S_2, S_3, S_4, S_5\}$. By the definition of an edge clique cover,

$$20 = |E(\overline{C_8})| \leq \sum_{i=1}^{5} |E(S_i)|.$$

Without loss of generality, we may assume that $|S_i| \geq |S_{i+1}|$ for $i = 1, 2, 3, 4$. Since $\{v_0, v_2, v_4, v_6\}$ and $\{v_1, v_3, v_5, v_7\}$ are the only cliques of maximum size, we have $|S_i| \leq 3$ for $i = 3, 4, 5$. If $|S_2| \leq 3$, then

$$\sum_{i=1}^{5} |E(S_i)| \leq 6 + 12 = 18,$$

which is a contradiction. Thus $|S_1| = |S_2| = 4$. Then we may assume that $S_1 = \{v_0, v_2, v_4, v_6\}$ and $S_2 = \{v_1, v_3, v_5, v_7\}$. No two of edges v_0v_3, v_1v_4, v_2v_7

belong to the same clique. Moreover none of them is covered by S_1 or S_2. Thus they must be covered by S_3, S_4, S_5. Without loss of generality, we may assume that $\{v_0, v_3\} \subseteq S_3$, $\{v_1, v_4\} \subseteq S_4$ and $\{v_2, v_7\} \subseteq S_5$. The edges v_3v_6, v_1v_6, v_0v_5, v_2v_5 still need to be covered. The edge v_1v_6 cannot be covered by S_3 or S_5 and so must be covered by S_4. Thus $S_4 = \{v_1, v_4, v_6\}$. Now the edge v_3v_6 must be covered by S_3 and so $S_3 = \{v_0, v_3, v_6\}$. Then v_0v_5 cannot be covered by any of $S_1, \ldots, S_5$, which is a contradiction. Thus $\theta_e(\overline{C_8}) \geq 6$. Furthermore,

$$\{v_0, v_3, v_5\}, \{v_2, v_5, v_7\}, \{v_4, v_1, v_7\}, \{v_6, v_1, v_3\}, \{v_0, v_2, v_4, v_6\}, \{v_1, v_3, v_5, v_7\}$$

form an edge clique cover of $\overline{C_8}$, and so $\theta_e(\overline{C_8}) = 6$.
(ii) Let n ($n \geq 9$) be an odd integer. We define the following sets:

$$S_1 := \{v_0, v_2, v_4, \ldots, v_{n-3}\};$$
$$S_2 := \{v_0, v_3, v_5, \ldots, v_{n-2}\};$$
$$S_3 := \{v_1, v_3, v_5, \ldots, v_{n-2}\};$$
$$T_1 := \{v_1, v_4, v_6, \ldots, v_{n-1}\};$$
$$T_3 := \{v_3, v_6, v_8, \ldots, v_{n-1}\};$$
$$T_5 := \{v_5, v_8, v_{10}, \ldots, v_{n-1}, v_2\};$$
$$T_i := \{v_i, v_2, v_4, \ldots, v_{i-3}, v_{i+3}, v_{i+5}, \ldots, v_{n-3}, v_{n-1}\} \quad (i = 7, \ldots, n-2, i: \text{ odd }).$$

It is easy to check that each of S_i $(i = 1, 2, 3)$ and T_j $(j = 1, 3, \ldots, n-2)$ is a clique of $\overline{C_n}$. Take any edge v_iv_j of $\overline{C_n}$ with $i < j$. We consider all the possible cases in the following:

- If both i and j are odd, then v_iv_j is covered by S_3.
- If i is odd and j is even, then $j - i \geq 3$ and v_iv_j is covered by T_i.
- Suppose that i is even and j is odd. Then $j - i \geq 3$. If $i = 0$, then v_iv_j is covered by S_2. If $i \neq 0$, then v_iv_j is covered by T_j.
- Suppose that both i and j are even. If $j \neq n-1$ then v_iv_j is covered by S_1. If $j = n-1$, then $i \neq 1$ and v_iv_j is covered by T_1 or T_5.

Therefore the family $\{S_1, S_2, S_3, T_1, T_3, T_5, \ldots, T_{n-2}\}$ is an edge clique cover of $\overline{C_n}$, and so $\theta_e(\overline{C_n}) \leq 3 + \frac{n-1}{2} = \frac{n+5}{2}$.
(iii) Let n ($n \geq 10$) be an even integer. We define the following sets:

$$\begin{aligned}
S &:= \{v_0, v_2, v_4, \ldots, v_{n-2}\};\\
T_0 &:= \{v_0, v_3, v_5, \ldots, v_{n-5}, v_{n-3}\};\\
T_2 &:= \{v_2, v_5, v_7, \ldots, v_{n-3}, v_{n-1}\};\\
T_i &:= \{v_i, v_1, \ldots, v_{i-3}, v_{i+3}, v_{i+5}, \ldots, v_{n-1}\} \quad (i = 4, 6, \ldots, n-4);\\
T_{n-2} &:= \{v_{n-2}, v_1, v_3, \ldots, v_{n-7}, v_{n-5}\}.
\end{aligned}$$

It is easy to see that S and T_i for $i = 0, 2, \ldots, n-2$ are cliques of $\overline{C_n}$. Take an edge v_iv_j of $\overline{C_n}$ with $i < j$. We consider all the possible cases in the following:

- If both i and j are even, then v_iv_j is covered by S.
- If i is even (resp. odd) and j is odd (resp. even), then $j - i \geq 3$ and v_iv_j is covered by T_i (resp. T_j).
- Suppose that both i and j are odd. If $j - i = 2$ or 4, then v_i and v_j both are adjacent to v_{j+3} since $n \geq 10$. If $j - i \geq 6$, then v_i and v_j both are adjacent to v_{i+3}. Thus there is an even integer k such that both v_iv_k and v_jv_k are edges of $\overline{C_n}$. This implies that v_iv_j is covered by T_k.

Therefore the family $\{S, T_0, T_2, T_4, \ldots, T_{n-2}\}$ is an edge clique cover of $\overline{C_n}$, and so $\theta_e(\overline{C_n}) \leq \frac{n}{2} + 1$. □

Now by Theorems 1.2 and 2.4, the following theorem immediately holds:

Theorem 2.5. *Let $\overline{C_n}$ be the complement of a cycle with n vertices and p be a positive integer.*

(i) *For $p \leq 4$, $\overline{C_6}$ is a p-competition graph.*
For $p \leq 1$, $\overline{C_7}$ is a p-competition graph.
For $p \leq 5$, $\overline{C_8}$ is a p-competition graph.

(ii) *If $n \geq 9$ is odd and $p \leq \frac{n-3}{2}$, then $\overline{C_n}$ is a p-competition graph.*

(iii) *If $n \geq 10$ is even and $p \leq \frac{n}{2}$, then $\overline{C_n}$ is a p-competition graph.*

3 Concluding Remarks

In this note, we gave a necessary and sufficient condition for a cycle C_n with n vertices and a sufficient condition for the complement of a cycle to be a p-competition graph. It would be interesting to give a necessary condition for the complement of a cycle to be a p-competition graph.

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